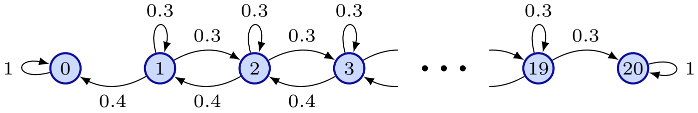
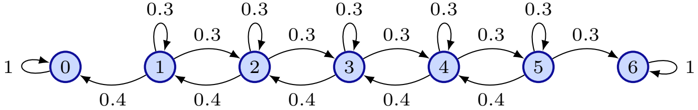
EE365: Structure of Markov Chains

Distribution propagation



- distribution propagation $\pi_{t+1} = \pi_t P$
- to find distribution of final states, compute $\pi_{\text{ss}} = \lim_{t \to \infty} \pi_t$
- called the *steady-state distribution*
- given by $\pi_{\text{ss}} = \pi_0 L$ where $L = \lim_{t \to \infty} P^t$

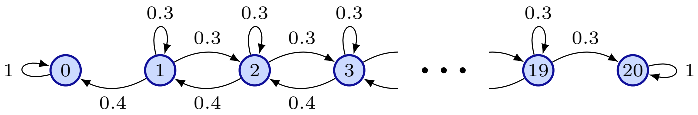
Example: absorption probabilities

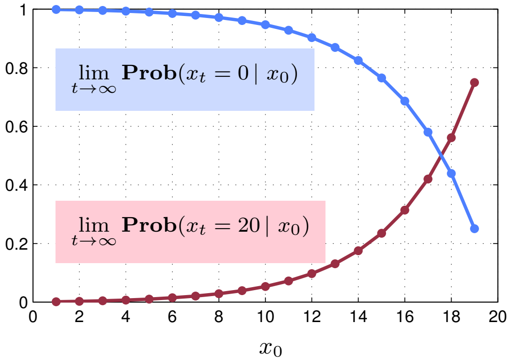


$$P = \begin{bmatrix} 0.3 & 0.3 & 0 & 0 & 0 & 0 & 0.4 \\ 0.4 & 0.3 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0.3 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.07 & 0.93 \\ 0 & 0 & 0 & 0 & 0 & 0.17 & 0.83 \\ 0 & 0 & 0 & 0 & 0 & 0.30 & 0.70 \\ 0 & 0 & 0 & 0 & 0 & 0.47 & 0.53 \\ 0 & 0 & 0 & 0 & 0 & 0.70 & 0.30 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

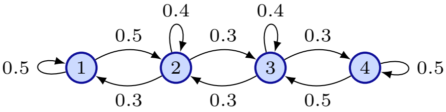
- $\pi_{\mathsf{ss}} = \pi_0 L$
- so initial state i leads to steady state distribution given by ith row of L,
- *e.g.*, L_{i6} is the probability of being captured by state 6 given $x_0 = i$

Example: absorption probabilities





Example: convergence



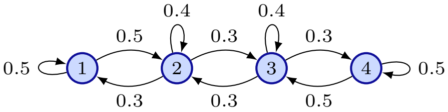
$$P = \begin{bmatrix} 0.5 & 0.5 & & \\ 0.3 & 0.4 & 0.3 & \\ & 0.3 & 0.4 & 0.3 \\ & & 0.5 & 0.5 \end{bmatrix}$$

- $L = \dfrac{1}{16} \begin{bmatrix} 3 & 5 & 5 & 3 \\ 3 & 5 & 5 & 3 \\ 3 & 5 & 5 & 3 \\ 3 & 5 & 5 & 3 \end{bmatrix}$
- in this case, L has the special form $L = \mathbf{1}\pi_{\mathsf{ss}}$
- π_t converges to π_{ss} from any initial state

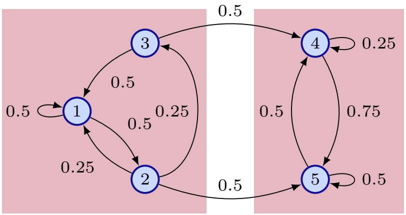
Irreducible matrices

P is called *irreducible* or *strongly connected* if

for every $i, j \in \mathcal{X}$ with $i \neq j$ there are paths $i \to j$ and $j \to i$



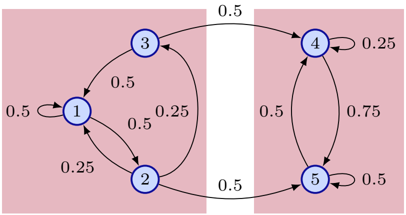
Irreducible components



$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ & & & 0.25 & 0.75 \\ & & & 0.5 & 0.5 \end{bmatrix}$$

- states can be grouped into *communicating classes*
- i, j are in the same class if there are paths $i \to j$ and $j \to i$
- a class with outgoing edges is called *transient*, otherwise it is called *recurrent* or *closed*

Irreducible components



$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0 & 0.5 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ & & & 0.25 & 0.75 \\ & & & 0.5 & 0.5 \end{bmatrix}$$

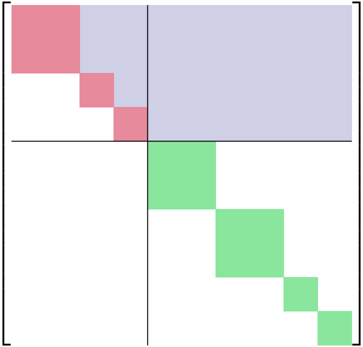
$$\lim_{t\to\infty} P^t = \begin{bmatrix} 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0.6 \end{bmatrix}$$

- $(\pi_t)_i \to 0$ for i in the transient class

General structure

every Markov chain can be decomposed as

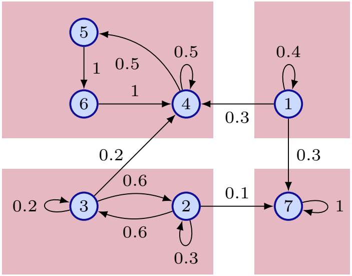
$$\begin{bmatrix} P_{11} & P_{12} \\ & P_{22} \end{bmatrix} =$$



- *transient classes:* P_{11} is block upper triangular, with irreducible blocks on the diagonal
- *closed classes:* P_{22} is block diagonal, with irreducible blocks

Irreducible components

for this example, P^t does not converge to the form $\mathbf{1}\pi$

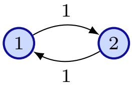


$$P = \begin{bmatrix} 0.4 & 0 & 0 & 0.3 & 0 & 0 & 0.3 \\ & 0.3 & 0.6 & 0 & 0 & 0 & 0.1 \\ & 0.6 & 0.2 & 0.2 & 0 & 0 & 0 \\ & & & 0.5 & 0.5 & 0 & \\ & & & 0 & 0 & 1 & \\ & & & 1 & 0 & 0 & \\ & & & & & & 1 \end{bmatrix}$$

$$\lim_{t\to\infty} P^t = \begin{bmatrix} 0 & 0 & 0 & 0.25 & 0.125 & 0.125 & 0.5 \\ 0 & 0 & 0 & 0.3 & 0.15 & 0.15 & 0.4 \\ 0 & 0 & 0 & 0.35 & 0.175 & 0.175 & 0.3 \\ 0 & 0 & 0 & 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0.5 & 0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pathological cases

there are pathological cases where neither P^t nor π_t converge, *e.g.*,



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Some results that we won't need ...

- P is called *regular* if $P^t > 0$ for some $t \geq 0$
- if P is irreducible, then P^t converges if and only if it is regular. Then

$$\lim_{t \to \infty} P^t = \mathbf{1}\pi_{\text{ss}}$$

- P^t converges iff every closed class is regular

Limit of powers

if P^t converges

$$L = \lim_{t\to\infty} P^t = \begin{bmatrix} 0 & (I - P_{11})^{-1} P_{12} L_{22} \\ 0 & L_{22} \end{bmatrix}$$

- $P_{11}^t \to 0$
- $P_{22}^t \to \mathbf{diag}\big(\mathbf{1}\pi_{\mathsf{inv}}^{(1)}, \ldots, \mathbf{1}\pi_{\mathsf{inv}}^{(k)}\big) = L_{22}$
- $PL = L$, hence

$$\begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} 0 & L_{12} \\ 0 & L_{22} \end{bmatrix} = \begin{bmatrix} 0 & L_{12} \\ 0 & L_{22} \end{bmatrix}$$

and so $P_{11} L_{12} + P_{12} L_{22} = L_{12}$ from which L_{12} is as above