# EE365: Shortest Paths

Deterministic optimal control

The simplest shortest path algorithm

Dijkstra's algorithm

# Deterministic optimal control

## Deterministic optimal control

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{T-1} g_t(x_t, u_t) + g_T(x_T) \\ \text{subject to} & x_{t+1} = f_t(x_t, u_t), \quad t = 0, \ldots, T-1 \end{array}$$

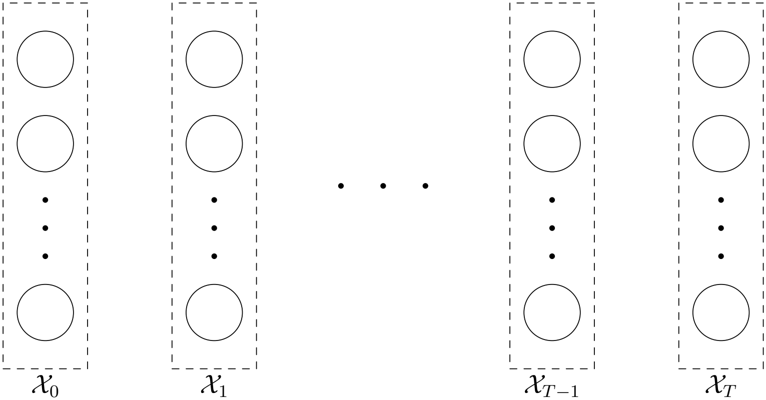
- variables are $x_1, \ldots, x_T$, $u_0, \ldots, u_{T-1}$. $x_0$ is given
- just an optimization problem, with a trivial information pattern
- can extend to case when costs are random, when dynamics are deterministic
- useful way to formulate many general optimization problems (*e.g.*, knapsack)

# Equivalent shortest path problems

create the *unrolled graph*

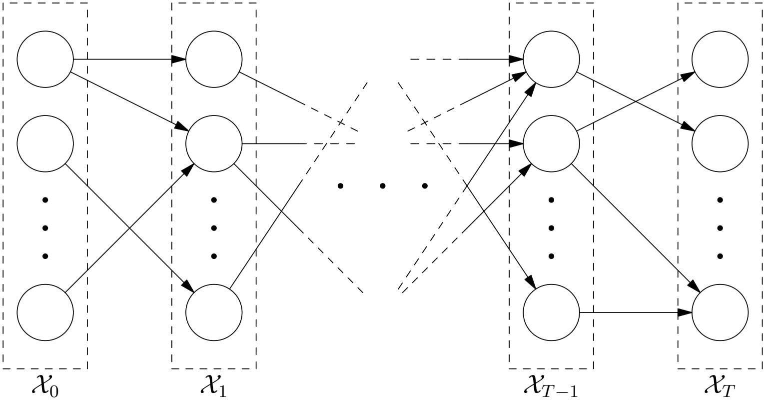
- *vertex set* is $\mathcal{V} = \mathcal{X}_0 \cup \cdots \cup \mathcal{X}_T$; if time-invariant, then $\mathcal{V} = \mathcal{X} \times \{0, \ldots, T\}$
- *directed edges* corresponding to $u_t$ from $x_t$ to $x_{t+1} = f_t(x_t, u_t)$
  if there are multiple edges, keep the lowest cost one
- *edge weights* are $g(x_t, u_t)$
- add additional *target vertex* $z$ with an edge from each $x \in \mathcal{X}_T$ with weight $g_T(x)$
- a sequence of actions is a path through the unrolled graph from $x_0$ to $z$
- associated objective is total, weighted path length

## Unrolled graph



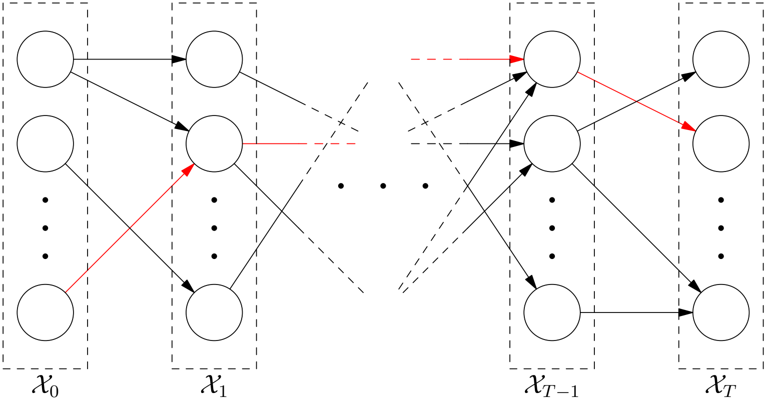

vertex set is $\mathcal{X}_0 \cup \cdots \cup \mathcal{X}_T$

## Unrolled graph




directed edges, labeled by $u_t$, from $x_t$ to $x_{t+1} = f_t(x_t, u_t)$

## Unrolled graph




a sequence of actions is a path through the unrolled graph

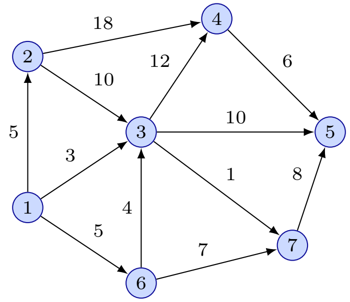
## Dynamic programming

- dynamic programming is often too computationally expensive
- $T|\mathcal{X}||\mathcal{U}|$ operations
- the *state space* can be so *large* that we cannot store the value function $v$
- specify the system in code by a function that returns $f(x, u)$ given $x, u$; called an *oracle* or an *implicit* description
- for MPC, we are only interested in finding the best action to take given the current state, not given any possible state (as given by DP)

# The simplest shortest path algorithm

## The simplest shortest path problem

- given weighted directed graph with vertices $\mathcal{V}$ and a source vertex $s \in \mathcal{V}$
- find lowest cost path from source to *every vertex*



## Problem description

- directed graph
- $\mathcal{V}$ is a finite set of vertices
- $\mathcal{E}$ is the set of directed edges $i \to j$
- for each $i$, $\mathcal{N}_i$ is the set of neighbors $j$ such that $i \to j$ is an edge
- $g_{ij}$ is the cost of edge $i \to j$
- $s$ is the source vertex
- $\mathcal{T} \subset \mathcal{V}$ is the target set
- $\mathbf{dist}(s, i)$ is the cost of the minimum-cost path from $s$ to $i$
- $\mathbf{dist}(s, \mathcal{T}) = \min_{i \in \mathcal{T}} \mathbf{dist}(s, i)$

# Problems with value iteration

- we have seen (one form of) the Bellman-Ford algorithm
- it finds the shortest path from a vertex $s$ to *all vertices*
- often we only want the shortest path from $s$ to some target set $\mathcal{T} \subset \mathcal{V}$
- *e.g.*, in the unrolled graph, $\mathcal{V} = \mathcal{X}_0 \cup \cdots \cup \mathcal{X}_T$, the source vertex is $x_0 \in \mathcal{X}_0$ and the target set is $\mathcal{T} = \{z\}$

## The simplest algorithm

$v_s = 0$
$v_i = \infty$ for all $i \neq s$
**while** there is an edge $i \to j$ such that $v_j > v_i + g_{ij}$
  let $i \to j$ be any such edge
  $v_j = v_i + g_{ij}$

- negative edge weights $g_{ij}$ allowed
- each step is called *edge relaxation*
- requires storing the array $v$, which is the size of $\mathcal{V}$
- finds the shortest path from source vertex $s$ to *all vertices*

# The simplest algorithm

$v_s = 0$
$v_i = \infty$ for all $i \neq s$
**while** there is an edge $i \to j$ such that $v_j > v_i + g_{ij}$
    let $i \to j$ be any such edge
    $v_j = v_i + g_{ij}$

if the graph has no negative cycles, then the algorithm terminates, because

- by induction, at every step $v_i$ is either $\infty$ or the cost of some path $s \rightsquigarrow i$
- these paths are always acyclic
- at every step, some $v_i$ decreases, and there are only finitely many paths

## The simplest algorithm

$v_s = 0$
$v_i = \infty$ for all $i \neq s$
**while** there is an edge $i \to j$ such that $v_j > v_i + g_{ij}$
    let $i \to j$ be any such edge
    $v_j = v_i + g_{ij}$

to show the algorithm terminates correctly, we will show that if there is no edge such that $v_j > v_i + g_{ij}$, then $v_i = \mathbf{dist}(s, i)$ for all $i$.

- suppose for a contradiction that $v_i \neq \mathbf{dist}(s, i)$ but $v_j = v_i + g_{ij}$ for all edges
- $v_i$ is the cost of some path $s \rightsquigarrow j \to k \rightsquigarrow i$
- let $j \to k$ be the first edge along the path such that $v_k > \mathbf{dist}(s, k)$
- then $v_j = \mathbf{dist}(s, j)$ and $\mathbf{dist}(s, k) \geq v_j + g_{jk}$, hence $v_k > v_j + g_{jk}$

## Properties of the simplest algorithm

- many well-known shortest path algorithms correspond to a particular choice of which order to relax edges
- often very fast
- one can construct (pathological) examples where it is very slow

# Dijkstra's algorithm

# Dijkstra's algorithm

$v_s = 0$
$v_i = \infty$ for all $i \neq s$
$F = \{s\}$
**while** $F \neq \emptyset$
  $i = \underset{i \in F}{\operatorname{argmin}}\, v_i$   // extract vertex $i$
  $F = F \setminus \{i\}$
  **for** $j \in \mathcal{N}_i$
    **if** $v_j > v_i + g_{ij}$
      $v_j = v_i + g_{ij}$
      $F = F \cup \{j\}$

- maintains a set $F$ called the *frontier* or *open* set
- terminates with $v_i = \mathbf{dist}(s, i)$ if graph has no negative cycles
- extracts the vertex with smallest $v_i$, relaxes its outgoing edges

## Dijkstra's algorithm

$v_s = 0$
$v_i = \infty$ for all $i \neq s$
$F = \{s\}$
**while** $F \neq \emptyset$
    $i = \operatorname{argmin}_{i \in F} v_i$    // extract vertex $i$
    $F = F \setminus \{i\}$
    **for** $j \in \mathcal{N}_i$
        **if** $v_j > v_i + g_{ij}$
            $v_j = v_i + g_{ij}$
            $F = F \cup \{j\}$

when all $g_{ij} \geq 0$

- algorithm extracts vertices in order of distance from $s$
- each vertex is extracted at most once
- $v_i \geq \mathbf{dist}(s, i)$ always; equality when $i$ is extracted

# Interpretation of Dijkstra's algorithm

- the algorithm may be thought of as a *simulation of fluid flow*
- imagine fluid traveling from the source vertex $s$, moving at speed 1
- $g_{ij}$ is time for fluid to traverse edge $i \to j$
- set $v_i$ at neighbors of $i$ to be the estimated time of arrival
- when fluid arrives at the next vertex, update the ETA of its neighbors
- some of these estimates may be too large, since the fluid might find shortcuts

## Keeping track of visited vertices

$v_s = 0$
$v_i = \infty$ for all $i \neq s$
$F = \{s\}$
$E = \emptyset$
**while** $F \neq \emptyset$
  $i = \operatorname{argmin}_{i \in F} v_i$  // extract vertex $i$
  $F = F \setminus \{i\}$
  $E = E \cup \{i\}$
  **for** $j \in \mathcal{N}_i$
    **if** $v_j > v_i + g_{ij}$
      $v_j = v_i + g_{ij}$
      $F = F \cup \{j\}$

- keeps track of $E$, the set of visited vertices, called the *closed* set

# Inductive proof

When all weights $g_{ij} \geq 0$, one can show by induction that, after each iteration

- there is some $d$ such that

$$\begin{aligned} \mathbf{dist}(s,i) &\leq d \qquad \text{for all } i \in E \\ \mathbf{dist}(s,i) &\geq d \qquad \text{for all } i \notin E \end{aligned}$$

- for all $i$, $v_i$ is the length of the shortest path $s \rightsquigarrow i$ *fully contained in* $E$

# Termination

$F = \{s\}$; $E = \emptyset$
$v_s = 0$
**while** $F \neq \emptyset$
    $i = \operatorname{argmin}_{i \in F} v_i$ // extract vertex $i$
1    $F = F \setminus \{i\}$; $E = E \cup \{i\}$
    **if** $i \in \mathcal{T}$ *terminate* // found target
    **for** $j \in \mathcal{N}_i$
        **if** $j \notin F \cup E$
            $v_j = v_i + g_{ij}$; $F = F \cup \{j\}$
        **else if** $j \in F$
            $v_j = \min\{v_j, v_i + g_{ij}\}$
        **else if** $v_j > v_i + g_{ij}$
            $v_j = v_i + g_{ij}$
2            $E = E \setminus \{j\}$; $F = F \cup \{j\}$ // removing from $E$ is optional

## Theorem

for any weights $g$ such that $\mathbf{dist}(i, \mathcal{T}) \geq 0$ for all $i \in \mathcal{V}$

- the algorithm terminates
- on termination, $v_i = \mathbf{dist}(s, i)$

- condition allows negative edges, but no negative cycles
- since $v_i$ is the optimal cost, assigning parents as the algorithm progresses gives a shortest path from $s$ to $i$
- note that the only reason we need additional assumptions (compared with the simplest algorithm) is that we are terminating the search early

## The closed set

- maintaining the set $E$ is optional
- the algorithm reduces to the previous one if we do not maintain $E$
- often $E$ is stored as a hash table, along with the values of $v$ in $E$
- if we remove from $E$ (in line 2), then $E$ and $F$ are always disjoint
- then (depending on the implementation) it may be easier to implement addition of elements to $E$ (in line 1)

## Efficient implementation

- store $F$ as a *heap* providing insert, delete, and extract-min operations
- since we terminate early, we do not need to store $v_i$ for every vertex $i$
- store $v$ using a *hash table*, or keep values of $v$ with vertices
- implement set $E$ as a hash table
- neither hash tables nor heaps are available in Matlab
- in Matlab arrays are a workaround, but scale poorly
- if $\mathcal{V}$ is small, then we can mark vertices as open/closed in an array instead of maintaining sets/lists

## Example: Two dimensional grid

- frontier $F$ shown in yellow
- closed set $E$ shown in blue